Square-Zero Extensions.

<u>Motivation:</u> let's discuss the notion of square-zero extensions for classical affine schemes and their relation to the cotangent space, i.e. module of Kähler differentials.

For $R \in CAlg^{disc.}$ a <u>square-zero extension</u> is a discrete commutative algebra R' w/ a surjective map:

$$\phi : R' \to R \quad \text{s.t.} \quad I^2 = 0, \quad I := \ker \phi.$$

Notice for any $M \in Mod_R^{\heartsuit}$ one has $R \oplus M \xrightarrow{\phi_M} R$ is a square-zero extension, which actually admits a splitting, i.e. $s : R \to R \oplus M$ s.t. $\phi_M \circ s = id_R$.

Given a square-zero extension by a module M, i.e. $M = I$ in the above one has:

$$(R \oplus M) \times_R R' \to R'$$

$$(r, m, r') \mapsto r' + m.$$

Informally, the set of sq-zero extensions of R by M is a torsor for $R \oplus M$ in $Ab(CAlg^{disc.}_{/R})$.

For $\phi : \tilde{R} \to R$ a sq-zero ext. s.t. $\ker \phi = M$.

<u>Lemma:</u> One has bijections:

$$\Omega_R = \Omega_{R/k}$$

$$\mathrm{Aut}_{CAlg^{disc.}_{/R}}(\tilde{R}) = \left\{ \begin{array}{c} \tilde{R} \xrightarrow{\alpha} \tilde{R} \\ \searrow \ \swarrow \\ R \end{array} \right\} \simeq \mathrm{Der}(R, M) \simeq \mathrm{Hom}_{Mod_R^{\heartsuit}}(\Omega_R, M).$$

Assuming that T^*S is perfect, one can check that:

$$H^{-i}(T^*S) \simeq Ext^i(T^*S, \mathcal{O}_S)^\vee . \qquad (\star).$$

Thus, when S is classical & f.t. one has $H^{-i}(T^*S) = 0$ for $i \geq 1$!

And we have. $T^* S_0 \simeq \Omega_{R_0}$, $S_0 = \mathrm{Spec}\, R_0$, $R_0 \in CAlg^{disc}$.

So one adds: $\mathrm{Aut}_{CAlg^{disc}_{/R}}(\tilde{R}) \simeq \mathrm{Hom}_{Mod_R^\heartsuit}(H^0(T^*\mathrm{Spec}\, R), M)$.

to the bijections of the previous Lemma.

Now $(\star)$ makes one wonder what is the role of $H^{-i}(T^*S)$ for $i \geq 1$ in terms of extensions of S.

Let's introduce some notation to deal w/ this question. For $S \in Sch^{aff}$ consider the category:

$$Sqz(S) := \left(QCoh(S)^{\leq -1}_{T^*S/-}\right)^{op}.$$

Notice that given $\gamma : T^*S \to \mathcal{F}$ one has

$S_{\mathcal{F}} \xrightarrow{\gamma^*} S_{T^*S}$. Also let $S_{T^*S} \xrightarrow{d} S$ denote the map induced by $id_{T^*(S)} \in \mathrm{Hom}_{QCoh(S)}(T^*(S), T^*(S))$.

Thus, we define $Sqz(T^*S \xrightarrow{\gamma} \mathcal{F}) := S \coprod_{pr\, S_{\mathcal{F}}} S_{\mathcal{F}}$ and the right map is the composite: $S_{\mathcal{F}} \xrightarrow{\gamma^*} S_{T^*S} \xrightarrow{d} S$.

Here is one way to think about $Sqz(S)$.

Let $Sch_{S/, inf\text{-}closed} := \{ f : S \to T \mid df^* : f^*T^*T \to T^*S$ s.t. $H^0(d^*f)$ is surjective, i.e. $T^*(S/T) \in QCoh(S)^{\leq -1} \}$.

The following is a consequence of the definitions, one has a pair of adjoint functors:

$$\mathrm{SqZ} : \left(\mathrm{QCoh}(S)^{\leq -1}_{T^*S/-}\right)^{op} \rightleftarrows \mathrm{Sch}_{S/,\,\text{inf-closed}}.$$

$$T^*S \to T^*(\mathbb{T}_{S/T}) \longleftarrow (T,\ f : S \to T)$$

$$\gamma : T^*S \to \mathcal{F} \longmapsto S \coprod_{S_{\mathcal{F}}} S.$$

? Warning: The functor $\mathrm{SqZ} : \left(\mathrm{QCoh}(S)^{\leq -1}_{T^*S/-}\right)^{op} \to \mathrm{Sch}_{S/}$

is <u>not</u> fully faithful.

The point being that in derived geometry, being a sq.-zero extension is not just a property but extra data, as one would normally expect b/c $\mathcal{I}^2 = 0$ when considered homotopically involves higher coherences not necessarily included in the map $S \to T$.

<u>Rk</u>: The notion of <u>n-small extension</u> remedies the above warning. Ⓐ

<u>Def'n</u>: A morphism $\mathrm{Spec}\, B = S \xrightarrow{f} T = \mathrm{Spec}\, A$ is an <u>n-small ext.</u> if $(\phi : A \to B)$

(i) $\mathrm{Fib}\,\phi \in \mathrm{Vect}^{\leq -2n}$ &

(ii) $\mathrm{Fib}\,\phi \otimes_A \mathrm{Fib}\,\phi \to \mathrm{Fib}\,\phi$ is homotopic to zero.

~~Notice that the connectivity estimate we have~~ ~~connectivity estimate~~

~~$f : S \to T$ n-small ext. $\Rightarrow$ $\mathrm{Cofib}\,\phi \in \mathrm{Vect}^{\leq -2n-1}$ $\Rightarrow$ $\mathrm{Cofib}(\phi) \otimes_A B \to \mathbb{L}_{B/A}$ is $2(2n+1)$-connective.~~ ~~$T^*(S/T)$ is~~

~~In particular, H^-~~

One can prove (see [HA, 7.4.1.23]) that one has an equivalence of categories

$$\{ f : S \to T \mid \text{n-small extension} \} \simeq \left(\mathrm{QCoh}(S)^{\geq -2n,\, \leq -1}_{T^*X/-}\right)^{op}.$$

http://www.plentysteps.com

We will address this failure of fully faithfulness by encoding this data slightly differently.

Def'n: For a fixed $\mathcal{F} \in QCoh(S)^{\leq -1}$ we will call

$$\mathrm{Hom}_{QCoh(S)}(T^*S, \mathcal{F})$$ the space of square-zero extensions of S by $I := \mathcal{F}[-1]$.

Here is the reason for this terminology. Suppose:

$$(S \hookrightarrow T)_{z} = S_{\mathcal{F}}\,z\,(T^*S \to \mathcal{F})$$ then one has a fiber seq:

$$z_*(I) \to \mathcal{O}_T \to z_*(\mathcal{O}_S)$$ and I plays the role of the "ideal" of definition of S inside T.

Notice the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{F} \in & (QCoh(S)^{\leq 0})^{op} & \longrightarrow & (QCoh(S)^{\leq -1}_{T^*X/-})^{op} & T^*S \xrightarrow{0} \mathcal{F}[1]. \quad \gamma : T^*S \to \mathcal{G}. \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ \begin{pmatrix} S \to S_{\mathcal{F}} \\ \downarrow \\ S \end{pmatrix} & Sch^{(aff)}_{S/-/S} & \longrightarrow & Sch^{(aff)}_{S/-} & S \coprod_{S_{\mathcal{G}}} S. \\ & \begin{pmatrix} S \to T \\ \downarrow \\ S \end{pmatrix} & \longmapsto & S \to T. & \end{array}$$

Indeed, we notice that since $S_{\mathcal{G}} \to S$, is a closed nil-isomorphism, i.e. ${}^{red}S_{\mathcal{G}} \xrightarrow{\sim} {}^{red}S$, and ${}^{cl}S_{\mathcal{G}} \hookrightarrow {}^{cl}S$.

The pushout

$$S \coprod_{S_{\mathcal{F}[1]}} S$$ can be performed in affine schemes:

$S = \mathrm{Spec}\,R$.

$\Gamma(S, \mathcal{F}) =: M$.

$$\mathrm{Spec}\Big(R \underset{R \oplus M[1]}{\times} R\Big) \simeq \mathrm{Spec}\,(R \oplus \underset{M[1]}{0 \times 0}) \simeq \mathrm{Spec}\,(R \oplus M) = \underline{S_{\mathcal{F}}}.$$

(classical to derived via sq.-zero extensions.)

Thm: (0) For $S \in {}^{cl}Sch^{aff.}$

$$\{\text{sq.-zero extensions of } S \text{ by } \mathcal{I} \in QCoh(S)^{\heartsuit}\} \simeq \{S \hookrightarrow S' \text{ closed in } {}^{cl}Sch^{aff} \text{ s.t. } \mathcal{I} \text{ of } S \text{ in } S' \text{ squares to } 0.\}$$

is i.e. $\left(QCoh(S)^{\leq -1, \geq -1}_{T^*S/-}\right)^{op}$

$n \geq 1$

(n) For $S_n \in {}^{\leq n}Sch^{aff.}$ The category:

$$\{S_{n+1} \in {}^{\leq (n+1)}Sch^{aff} \mid {}^{\leq n}S_{n+1} \simeq S_n\} \simeq \{\text{sq.-zero ext. of } S_n \text{ by } \mathcal{I}_n \in QCoh(S)^{\heartsuit}[n+1]\}$$

Pf: Consider $\imath: S_n \hookrightarrow S_{n+1}$ this gives:

$$\imath_*(\mathcal{F}[-1]) \to \mathcal{O}_{S_{n+1}} \to \imath_* \mathcal{O}_{S_n} \quad \text{for } \mathcal{F} \in QCoh(S_n)^{\heartsuit}[n+2].$$

Claim: $\exists$ a map $\gamma: T^*(S_n) \to \mathcal{F}$ s.t. $S_{n+1} \simeq S_n \coprod_{(S_n)_{\mathcal{F}}} S_n$. $\lrcorner$

Consider the fiber sequence:

$$\imath^* T^*(S_{n+1}) \xrightarrow{(d\imath)^*} T^*(S_n) \to T^*(S_n/S_{n+1}).$$

Claim 2: $-\ H^k(T^*(S_n/S_{n+1})) = 0$ for $k \geq -n-1$

$-\ H^{-n-2}(T^*(S_n/S_{n+1})) \simeq \mathcal{F}$.

These are consequences of the connectivity estimates.

For (0), one has that for ϕ $R' \to R$ Cfib ϕ is 1-connective, so $R' \otimes_R \text{Cfib}(\phi) \to \mathbb{L}_{R'/R}$ is 2-connective, i.e.

$\mathcal{I}[1] \to T^*(S'/S)$ induces an isom. on $Mod_{R'}^{\geq -1, \leq 0}$

$\Rightarrow H^{-1}(T^*(S'/S)) \simeq H^0(\mathcal{I}). (\simeq \mathcal{I}.)$

For (n) we have:

$$z_*(\mathcal{F}[-1])[1] \to T^*(S_n/S_{n+1}) \quad \text{is an isom. on } QCoh(S_n)^{\geq -n-1,\leq 0}.$$

i.e. $H^k(T^*(S_n/S_{n+1})) \simeq 0$ for $k \geq -n-1$.

And, $H^{-n-2}(T^*(S_n/S_{n+1})) = \mathcal{F}$. ∎

Rk: The above discussion of square-zero extensions makes sense for schemes as well. Whereas the initial input is that for any scheme $X \in Sch$, one has

$$T^*_x X \in QCoh(S)^{\leq 0} \quad \text{for any } S \to X \quad b/c.$$

$$\begin{array}{ccc} Maps_{S/-}(S_{\mathcal{F}}, X) & \to & Maps_{S/-}(S_{\mathcal{F}_2}, X) \underset{Maps_{S/-}(S_{\mathcal{F}_1}, X)}{\times} * \quad \text{is} \\ \text{is} & & \\ Hom_{Sch}(S_{\mathcal{F}}, X) \underset{Hom_{Sch}(S,X)}{\times} * & \to & \left(Hom_{Sch}(S_{\mathcal{F}_2}, X) \underset{Hom_{Sch}(S_{\mathcal{F}_1}, X)}{\times} Hom_{Sch}(S, X) \right) \underset{Hom_{Sch}(S,X)}{\times} * \end{array}$$

& $S_{\mathcal{F}_2} \underset{S_{\mathcal{F}_1}}{\amalg} S \simeq S$ in Sch.

Rk: One can rephrase the theorem above as follows:

One has a fully faithful (right adjoint) functor.

$${}^{\leq(n+1)}Sch^{aff} \to SqZ(Sch)^{aff}_{\leq 0} \underset{(Sch^{aff})^2}{\times} {}^{\leq n}Sch^{aff} \times {}^{\leq n+1}Sch^{aff} \quad , \text{ where.}$$

$$SqZ(Sch^{aff}) := \{ S \in Sch^{aff} \mid \gamma : T^*(S) \to \mathcal{F},\ \mathcal{F} \in QCoh(S)^{\leq -1} \}$$

and morphisms $f: S \to T$

$$\begin{array}{c} T^*(T) \to f_* T^*(S) \to f_* \mathcal{F}. \\ \searrow \quad \downarrow \alpha \\ \mathcal{G} \end{array}$$